TORSION OF HOLLOW CYLINDERS

BY

R. C. F. BARTELS

1. **Introduction.** The torsion problem for the (solid) cylinder whose cross section is a simply connected region has received considerable attention in recent literature. Outstanding among the published works which emphasize methods are the investigations by: Trefftz [9] and the generalizations of his method by Seth [7]; Muschelisvilli [5] and the applications of his method by Sokolnikoff and Sokolnikoff [7]; and more recently Stevenson [8] and the extension of his method by Morris [4].

The torsion problem for the (hollow) cylinder whose cross section is a doubly connected region, on the other hand, has not enjoyed such propitious attention. The present analytical methods of treating this form of the problem have been improved very little since the close of the nineteenth century when Macdonald [3] obtained a solution for the region bounded by eccentric circles making use of curvilinear orthogonal coordinates; the solution of the torsion problem for the region bounded by confocal ellipses was published by Greenhill [1] several years earlier employing the same method. It should be remarked that the experimental methods—for example, the membrane analogy which was pointed out by Prandtl [12] and later improved by Griffith and Taylor [13], and Trayer and March [14]—are readily extended to the case of multiply connected regions.

The purpose of this paper is twofold: first, to supply the need for a general method of obtaining a computable solution of boundary value problems of Dirichlet type for doubly connected regions, and second, to apply this method to obtain the solution of the torsion problem for certain hollow cylinders.

The procedure of determining the solutions of the torsion problem for the doubly connected regions considered is in each case to map the region conformally upon an annulus, and then to solve the related Dirichlet problem for the simpler region. To this end a formula for the solution of the general Dirichlet problem for the annulus is developed which, though lacking the elegance of the well known integral formula of Villat [10], lends itself readily for purposes of computation.

2. **Solution of the problem for the annular region.** Let γ_1 and γ_2 denote the circles $|\zeta| = r_1$, and $|\zeta| = r_2$, $r_1 < r_2$, respectively, in the plane of the complex variable ζ. Also, let the *real* functions $u_1(\sigma_1)$ and $u_2(\sigma_2)$, where $\sigma_1 = r_1 e^{i\phi}$, $\sigma_2 = r_2 e^{i\phi}$ (ϕ real), be periodic and continuous for all values of ϕ with period 2π and such that

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$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{u_j(\sigma_j)}{\sigma_j}d\sigma_j=\frac{1}{2\pi}\int_0^{2\pi}u_j(\sigma_j)d\phi=A \qquad (j=1,2), \tag{1}$$

A representing the common value of the integrals. Then the function(1)

$$f(\zeta)=\frac{1}{\pi i}\int_{\gamma_2}\frac{U_2(\sigma_2)}{\sigma_2-\zeta}d\sigma_2-\frac{1}{\pi i}\int_{\gamma_1}\frac{U_1(\sigma_1)}{\sigma_1-\zeta}d\sigma_1+\text{const.}, \tag{2}$$

where the functions $U_1(\sigma_1)$ and $U_2(\sigma_2)$ are defined by the integral equations(2)

$$u_j(\sigma_j)=U_j(\sigma_j)+\Re\left\{\frac{1}{\pi i}\int_{\gamma_k}\frac{U_k(\sigma_k)}{\sigma_2-\sigma_1}d\sigma_k\right\} \qquad (j,k=1,2;\,j\neq k), \tag{3}$$

is single-valued and regular for $r_1<|\zeta|<r_2$ and, except for an additive constant, its real part takes on the values $u_1(\sigma_1)$, $u_2(\sigma_2)$ on the circles γ_1, γ_2, respectively. The existence of the functions U_1, U_2 has been established for functions u_1, u_2 satisfying much more general conditions than those considered here. It is well known that the condition (1) is both necessary and sufficient in order that the function $f(\zeta)$ determined by the values $u_1(\sigma_1)$ and $u_2(\sigma_2)$ prescribed on γ_1 and γ_2, respectively, be single-valued.

If the functions u_1 and u_2 are replaced by two new functions satisfying (1) and differing from u_1 and u_2 by constants, the function $f(\zeta)$ is altered only by the addition of a constant. It can therefore be assumed, without restricting the application of the formula (2), that the constant A in (1) is zero. In this event,

$$\int_0^{2\pi}U_j(\sigma_j)d\phi=0 \qquad (j=1,2). \tag{4}$$

In addition to the conditions given above, let $u_1(\sigma_1)$ and $u_2(\sigma_2)$ be absolutely continuous functions of ϕ in the interval $0\leqq\phi\leqq 2\pi$. Then $U_1(\sigma_1)$, $U_2(\sigma_2)$ are also absolutely continuous functions of ϕ in the same interval. Under these conditions the infinite series

$$u_j(\sigma_j)=\sum_{m=1}^{\infty}\left[\overset{(j)}{a}_m\sigma_j^m+\overset{(j)}{a}_{-m}\sigma_j^{-m}\right],\quad U_j(\sigma_j)=\sum_{m=1}^{\infty}\left[\overset{(j)}{A}_m\sigma_j^m+\overset{(j)}{A}_{-m}\sigma_j^{-m}\right] \qquad (j=1,2); \tag{5}$$

where

$$\overset{(j)}{a}_m=\frac{1}{2\pi i}\int_{\gamma_j}\frac{u_j(\sigma_j)}{\sigma_j^{m+1}}d\sigma_j,\qquad \overset{(j)}{A}_m=\frac{1}{2\pi i}\int_{\gamma_j}\frac{U_j(\sigma_j)}{\sigma_j^{m+1}}d\sigma_j \qquad (j=1,2;\,m=1,2,\cdots), \tag{6}$$

(1) Cf. G. C. Evans, *The logarithmic potential*, Amer. Math. Soc. Colloquium Publications, vol. 6, 1927, pp. 112–117.

(2) The symbol $\Re\{F\}$ is understood to mean the real part of F.

converge uniformly for all values of ϕ; the constant terms corresponding to $m=0$ are absent from these series as a consequence of equations (1) and (4). Therefore, by equations (2) and (5), it follows that, for $r_1<|\zeta|<r_2$,

$$(7)\qquad f(\zeta) = 2\sum_{m=1}^{\infty} A_m^{(2)}\zeta^m + 2\sum_{m=1}^{\infty} A_{-m}^{(1)}\zeta^{-m} + \text{const.}$$

Also, on substituting the series (5) in (3) and equating coefficients, it is seen that

$$(8)\qquad A_m^{(2)} = \frac{r_2^{2m}a_m^{(2)} - r_1^{2m}a_m^{(1)}}{r_2^{2m} - r_1^{2m}}, \qquad A_m^{(1)} = \frac{r_2^{2m}a_{-m}^{(1)} - r_1^{2m}a_{-m}^{(2)}}{r_2^{2m} - r_1^{2m}}.$$

Since $r_1<r_2$, the coefficients in (8) can be written in the form of absolutely convergent infinite series as follows:

$$A_m^{(2)} = a_m^{(2)} + [a_m^{(2)} - a_m^{(1)}]\sum_{n=1}^{\infty} p_n^m, \qquad A_{-m}^{(1)} = a_{-m}^{(1)} + [a_{-m}^{(1)} - a_{-m}^{(2)}]\sum_{n=1}^{\infty} p_n^m,$$

where

$$(9)\qquad p_n = (r_1/r_2)^{2n} < 1.$$

Let these be substituted in the infinite series in (7), which also converges absolutely when $r_1<|\zeta|<r_2$. Then, with the aid of the first of equations (6) and equations (1) in which $A=0$, and by interchanging the order of summation with respect to m and n, it follows, after rearranging terms, that

$$(10)\quad f(\zeta) = \frac{1}{\pi i}\sum_{n=-\infty}^{\infty}\left[\int_{\gamma_2}\frac{u_2(\sigma_2)}{\sigma_2 - p_n\zeta}\,d\sigma_2 - \int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1 - p_n\zeta}\,d\sigma_1\right] + \text{const.}$$

whenever $r_1<|\zeta|<r_2$.

The regularity of the function $f(\zeta)$ defined by the infinite series in (10) is easily established under less restrictive conditions on the functions $u_1(\sigma_1)$ and $u_2(\sigma_2)$ with the aid of the following interesting lemma:

LEMMA. *Let $F(z)$ be regular and $|F(z)|<M$ for $|z|<r$, and let $F(0)=0$. Then the infinite series $\sum_{n=1}^{\infty}F(q^n z)$, where $0<q<1$, converges uniformly and absolutely for $|z|\leqq r$ and, consequently, defines an analytic function which is regular for $|z|\leqq r$. If $F(z)$ is regular and $|F(z)|<M$ for $|z|>r$, and if $F(\infty)=0$, then the same conclusions hold for the infinite series $\sum_{n=1}^{\infty}F(q^n z)$, where $q>1$, when $|z|\geqq r$.*

The lemma follows at once from the lemma of Schwarz[3]. For, applying the latter,

[3] Cf. E. F. Titchmarsh, *The theory of functions*, 2d edition, London, 1939, p. 168.

$$\sum_{n=1}^{\infty}|F(q^n z)| \leqq \sum_{n=1}^{\infty}\frac{M|z|}{r}q^n \leqq M\sum_{n=1}^{\infty}q^n = \frac{Mq}{1-q}$$

whenever $|z| \leqq r$. Hence the series converges absolutely and uniformly if $|z| \leqq r$. The second part of the lemma is proved in like manner.

Equation (10) can be written in the form

$$(11) \qquad f(\zeta) = \frac{1}{\pi i}\int_{\gamma_2}\frac{u_2(\sigma_2)}{\sigma_2-\zeta}d\sigma_2 - \frac{1}{\pi i}\int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1-\zeta}d\sigma_1 + \Sigma(\zeta) + \text{const.},$$

where $\Sigma(\zeta)$ represents the infinite series obtained from that in (10) by omitting the term corresponding to $n=0$. The application of the lemma to the infinite series $\Sigma(\zeta)$ is at once evident with the aid of the following inequalities which, in view of the inequality (9), are seen to hold for $n \geqq 1$ and $r_1 \leqq |\zeta| \leqq r_2$:

$$(12) \qquad |p_{-n}| > 1, \qquad |p_n\zeta| < r_1, \qquad |p_{-n}\zeta| > r_2.$$

If $u_1(\sigma_1)$ and $u_2(\sigma_2)$ are merely bounded and integrable functions of ϕ in the interval $(0, 2\pi)$, the terms of the infinite series $\Sigma(\zeta)$ for $n \geqq 1$ are bounded and regular when $|\zeta| \leqq r_2$ and, as a consequence of equations (1) with $A=0$, vanish for $\zeta=0$. The terms corresponding to $n \leqq -1$ are bounded and regular for $|\zeta| \geqq r_1$ and are seen to vanish for $\zeta=\infty$. Therefore by the lemma the series $\Sigma(\zeta)$ converges absolutely and uniformly for $r_1 \leqq |\zeta| \leqq r_2$. Hence the function $\Sigma(\zeta)$ is regular for $r_1 < |\zeta| < r_2$ and continuous on the circles γ_1 and γ_2.

It will now be shown that the real part of the function $f(\zeta)$ defined in the equation (10) takes on appropriate values on the circles γ_1 and γ_2. For, if $\zeta_2 = r_2 e^{i\theta}$ represents an arbitrary value of ζ on γ_2, it is readily verified using equation (1) with $A=0$ and elementary manipulations that

$$\begin{aligned}\overline{\Sigma(\zeta_2)} &= -\Sigma(\zeta_2) + \frac{1}{\pi i}\int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1-\zeta_2}d\sigma_1 - \frac{1}{\pi i}\int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1-p_1\zeta_2}d\sigma_1,\\ &= -\Sigma(\zeta_2) + 2\Re\left\{\frac{1}{\pi i}\int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1-\zeta_2}d\sigma_1\right\},\end{aligned}$$

where the bar denotes the conjugate values. Therefore

$$\Re\{\Sigma(\zeta_2)\} = \Re\left\{\frac{1}{\pi i}\int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1-\zeta_2}d\sigma_1\right\}.$$

As a consequence of this, it follows from (11) that

$$\lim_{\zeta\to\zeta_2}\Re\{f(\zeta)\} = \lim_{\zeta\to\zeta_2}\Re\left\{\frac{1}{\pi i}\int_{\gamma_2}\frac{u_2(\sigma_2)}{\sigma_2-\zeta}d\sigma_2\right\} + \text{real const.}$$

Under the assumption that $u_2(\sigma_2)$ be bounded and integrable, the limit on the right-hand side of the last equation exists and is equal to $u_2(\sigma_2)$ if ζ_2 is a point of continuity of $u_2(\sigma_2)$(4).

The behavior of the function $f(\zeta)$ on the circle γ_1 can be studied in like manner. The results of the foregoing investigation can be stated as follows(5): *Let the functions $u_1(\sigma_1)$ and $u_2(\sigma_2)$, satisfying* (1) *with $A=0$, be bounded and integrable with respect to ϕ in the interval* $(0, 2\pi)$. *Then the analytic function $f(\zeta)$ which is single-valued and regular in the annular region between the circles γ_1 and γ_2 and whose real part takes on the values $u_1(\sigma_1)$ and $u_2(\sigma_2)$ at all points of γ_1 and γ_2, respectively, at which these functions are continuous is given by the formula in equation* (10).

Formula (10) will be seen to form the basis for a very general method of treating a class of boundary value problems related to doubly connected regions. In the following sections this formula is applied to the problem of the torsion of hollow cylinders.

3. **Statement of the torsion problem.** In the theory of elasticity the St. Venant's torsion problem for a region D of the xy-plane may be formulated as that of determining an analytic function $F(z)$ of the complex variable $z=x+iy$ which is single-valued and regular within D and such that at points of its boundary C(6)

$$2\Re\{F(z)\} = x^2 + y^2 + \text{const.} \tag{13}$$

Given the function $F(z)$, the important physical quantities for a twisted, homogeneous, cylindrical beam whose cross section has the shape of the region D can be determined. For example, the well known formulas for the shearing stresses X_z and Y_z at a point in any cross section can be written in the form

$$Y_z + iX_z = \tau G[\bar{z} - F'(z)],$$

where τ is the "twist," and G is the slide modulus of the material constituting the beam. Its torsional rigidity J is readily written in the form

$$J = GI_0 - \frac{iG}{4}\int_C [F(z) - \overline{F(z)}]d(x^2 + y^2),$$

where I_0 is the moment of inertia of the region D with respect to the origin, and the integral is taken over the complete boundary C. Making use of (13) and the fact that $F(z)$ is single-valued, the latter can be written

$$J = GI_0 + \frac{G}{2}\Re\left\{i\int_C \overline{F(z)}dF(z)\right\}.$$

(4) Evans, loc. cit., pp. 39, 65.
(5) Compare with Villat [10].
(6) Cf. Love [2, p. 314].

In the present paper D is taken as a doubly connected region bounded internally and externally by the closed Jordan curves C_1 and C_2, respectively. In view of the significance of equation (1) of the preceding section, it is evident that the values of the constants in (13) on the curves C_1 and C_2 are not independent; otherwise the function $F(z)$ determined by the boundary condition is not necessarily single-valued in D. However, apart from this restriction, the constants are arbitrary.

Let the function $z=\omega(\zeta)$ map D one-one and conformally on the interior of an annular region bounded by two circles $|\zeta|=r_1$ and $|\zeta|=r_2$, $r_1<r_2$, in the plane of the complex variable ζ. As before, these circles are denoted by γ_1 and γ_2 and the values of ζ on them by $\sigma_1=r_1e^{i\phi}$ and $\sigma_2=r_2e^{i\phi}$, respectively. It is understood, of course, that the radii of the circles γ_1 and γ_2 are not independent, the ratio r_1/r_2 being determined uniquely by the region D(7). Moreover the mapping is known to be topological on the boundary, so that the function $\omega(\zeta)$ is continuous on γ_1 and γ_2.

Given the mapping function $z=\omega(\zeta)$, the torsion problem for the doubly connected region D can be transformed into a corresponding boundary value problem for an annulus whose solution is given in the foregoing section. For, if $f(\zeta)$ represents the values of $F(z)$ in the ζ-plane, that is, $f(\zeta)=F[\omega(\zeta)]$, then $f(\zeta)$ is single-valued and regular for $r_1<|\zeta|<r_2$ and, according to (13), its real part satisfies the conditions

$$\mathfrak{R}\{f(\sigma_j)\} = u_j(\sigma_j) = \frac{\omega(\sigma_j)\overline{\omega(\sigma_j)}}{2} + c_j \qquad (j = 1, 2), \tag{14}$$

where the constants c_1, c_2 are taken so that equations (1) with $A=0$ are satisfied. Since $\omega(\zeta)$ is continuous on γ_1 and γ_2, the functions $u_1(\sigma_1)$, $u_2(\sigma_2)$ are continuous on γ_1, γ_2, respectively. Thus the function $f(z)$ satisfies the conditions of the theorem stated at the close of the preceding section and, therefore, is given by the formula (10).

The function $F(z)$ representing the solution of the torsion problem for the region D is obtained from $f(\zeta)$ by making the inverse of the transformation $z=\omega(\zeta)$. On the other hand, this process of inversion is more or less superfluous since the important physical quantities for the cylindrical beam with cross section D can easily be expressed in terms of the function $f(\zeta)$. Thus the shearing stress at a point z of the cross section is given by

$$Y_z + iX_z = \tau G\left[\overline{\omega(\zeta)} - \frac{f'(\zeta)}{\omega'(\zeta)}\right],$$

where ζ is the point of the annular region in the ζ-plane corresponding to z

(7) For particulars on the mapping of multiply connected regions, see C. Carathéodory, *Conformal representations*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 28, London, 1932, pp. 70–73.

under the transformation $z=\omega(\zeta)$. Moreover the torsional rigidity J is given by

$$J = GI_0 + \frac{G}{2}\Re\left\{ i \int \overline{f(\zeta)} df(\zeta) \right\},$$

where the integral is taken over the complete boundary of the annulus in the positive sense.

4. **Torsion of the eccentric ring.** Let

$$z = \omega(\zeta) \equiv \frac{\zeta + a}{1 + a\zeta}, \tag{15}$$

where $a<1$. Then the region D of the z-plane corresponding to the annular region $r<|\zeta|<1$ of the ζ-plane is that which is bounded internally and externally by the circles $|z-c|=R$ and $|z|=1$, respectively, where

$$c = \frac{a(1 - r^2)}{1 - a^2r^2}, \qquad R = \frac{r(1 - a^2)}{1 - a^2r^2}.$$

This mapping of the region D upon the annulus is such that the points of the circles $\gamma_1(|\zeta|=r)$ and $\gamma_2(|\zeta|=1)$ correspond, respectively, to points of the circles $|z|=1$ and $|z-c|=R$.

Since in this case $\sigma_2\bar{\sigma}_2=1$, it follows that $\omega(\sigma_2)\overline{\omega(\sigma_2)}=1$. Also, by the theory of residues, it is seen that

$$\int_{\gamma_1} \frac{\omega(\sigma_1)\overline{\omega(\sigma_1)}}{\sigma_1} d\sigma_1 = 2\pi i(1 - h),$$

where

$$h = \frac{(1 - a^2)(1 - r^2)}{(1 - a^2r^2)}. \tag{16}$$

Accordingly, the functions $u_1(\sigma_1)$ and $u_2(\sigma_2)$ in the boundary condition (14) are given by

$$u_1(\sigma_1) \equiv [\omega(\sigma_1)\overline{\omega(\sigma_1)} + h - 1]/2, \qquad u_2(\sigma_2) \equiv 0. \tag{17}$$

It follows from the second of equations (17) that the first integral of each term of the infinite series in (10) vanishes. Further, with due regard for the inequalities in (12) in which $p_n=r^{2n}$, a simple evaluation of residues gives, for $r<|\zeta|<1$,

$$\int_{\gamma_1} \frac{u_1(\sigma_1)}{\sigma_1 - p_n\zeta} d\sigma_1 = \begin{cases} \dfrac{ahp_n\zeta}{1 + ap_n\zeta} & \text{when} \quad n > 0, \\[2ex] -\dfrac{ah}{a + p_{n-1}\zeta} & \text{when} \quad n \leqq 0. \end{cases}$$

Therefore, in accordance with (10), the function $f(\zeta)$ which is regular for $r<|\zeta|<1$ and satisfies the condition (14) on the circles γ_1 and γ_2 can be written in the form

$$f(\zeta) = h[L(\zeta) - L(1/\zeta)] + \text{const.}, \tag{18}$$

where,

$$L(\zeta) = \sum_{n=1}^{\infty} \frac{ar^{2n}}{\zeta + ar^{2n}} \cdot$$

By a transformation(8) of the Lambert series $L(\zeta)$ in (18), the function $f(\zeta)$ can also be written in the form(9)

$$f(\zeta) = \frac{ahr^2}{ar^2 + \zeta} + h[P(\zeta/r) - P(r/\zeta)] + \text{const.} \quad (r < |\zeta| < 1), \tag{19}$$

where

$$P(t) = \sum_{n=1}^{\infty} (-1)^n \frac{a^n r^{3n}}{1 - r^{2n}} t^n.$$

The latter form has the advantage over the former from the point of view of computation.

5. **Torsion of a hollow lune.** Let

$$z = \omega(\zeta) \equiv \frac{1 - (1 - \zeta^2)^{1/2}}{\zeta} \qquad (|\zeta| < 1), \tag{20}$$

where that branch of the square root is chosen which has the value $+1$ when $\zeta=0$. Then the annulus $r<|\zeta|<1$ of the ζ-plane corresponds to the region D of the z-plane which is bounded externally by arcs of equal radii intersecting at right angles in the points $z=\pm 1$, and internally by the oval

$$r^2[(x^2 + y^2 - 1)^2 - 4y^2] = 4(1 - r^2)(x^2 + y^2).$$

The region D is an approximation of the cross section of a common type of hollow strut used in aircraft construction. By choosing values of r sufficiently near unity, the comparison can be extended to thin cylindrical shells whose sections are in the shape of the two intersecting circular arcs determining the external boundary of D.

If as in the preceding section $\sigma_1 = re^{i\phi}$, it follows from (20) and the theory of residues that, for $r>0$,

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(8) Cf. K. Knopp, *Theory and application of infinite series*, English translation by R. C. Young, London, 1928, p. 452.

(9) The form of the solution given by (19) corresponds to that which was obtained by Weinel with the aid of dipolar coordinates; see [11, p. 70]. The method employed in the present paper is certainly more direct than that used by Weinel. Macdonald's [3] form of the solution can be found in Love [2, p. 320].

$$(21)\quad \frac{r^2}{2\pi i}\int_{\gamma_1} \frac{\omega(\sigma_1)\overline{\omega(\sigma_1)}}{\sigma_1-\mu}\,d\sigma_1 = \frac{1}{\pi}J(\mu) + \begin{cases} -(1-\mu^2)^{1/2} & \text{when } |\mu|<r, \\ 1-r^4/\mu^2)^{1/2}-1 & \text{when } |\mu|>r, \end{cases}$$

where

$$J(\mu) = \frac{1}{2i}\int_{\gamma_1} \frac{((1-\sigma_1^2)(1-\bar{\sigma}_1^2))^{1/2}}{\sigma_1-\mu}\,d\sigma_1;$$

these integrals are taken around the circle γ_1 ($|\zeta|=r$). The integral $J(\mu)$ can evidently be written in the form

$$(22)\qquad J(\mu) = r(1+r^2)\int_0^\pi \frac{r-\mu\cos\phi}{(r^2+\mu^2)-2r\mu\cos\phi}(1-k_1^2\cos^2\phi)^{1/2}d\phi,$$

where

$$k_1 = \frac{2r}{1+r^2}\cdot$$

Therefore(10), if $0<r<1$,

$$J(\mu) = (1+r^2)E_1 + (1+r^2)\frac{r^2-\mu^2}{r^2+\mu^2}K_1$$

$$(23)\qquad + \begin{cases} -K_1\dfrac{((1-\mu^2)(r^4-\mu^2))^{1/2}}{\mu}Z(\alpha, k_1) & \text{for } |\mu|<r, \\ K_1\dfrac{((\mu^2-1)(\mu^2-r^4))^{1/2}}{\mu}Z(\alpha, k_1) & \text{for } |\mu|>r, \end{cases}$$

where K_1 and E_1 are the complete elliptic integrals of the first and second kinds, respectively, α is the elliptic integral of the first kind defined as follows

$$(24)\qquad \operatorname{sn}(\alpha, k_1) = \frac{\mu(1+r^2)}{r^2+\mu^2},$$

and $Z(\alpha, k_1)$ is the Jacobi zeta-function; these elliptic integrals all have the modulus k_1.

In particular, if $\mu=0$, equations (21) and (22) give

$$\frac{1}{2\pi i}\int_{\gamma_1}\frac{\omega(\sigma_1)\overline{\omega(\sigma_1)}}{\sigma_1}\,d\sigma_1 = \frac{1}{\pi r^2}[2(1+r^2)E_1-\pi],$$

and consequently the constant c_1 in the definition of the function $u_1(\sigma_1)$ in (14) becomes

(10) For particulars on the evaluation of the integral $J(\mu)$, see E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th edition, London, 1935, pp. 499, 518, 522.

$$c_1 = \frac{1}{2\pi r^2}\,[\pi - 2(1 + r^2)E_1]. \tag{25}$$

Making use of an elementary quadratic transformation[11], the complete elliptic integrals K_1 and E_1 with modulus $k_1 = 2r/1+r^2$ can be expressed in terms of the complete elliptic integrals K and E with modulus r^2 as follows

$$K_1 = (1 + r^2)K \quad \text{and} \quad E_1 = \frac{2E}{1 + r^2} - (1 - r^2)K.$$

Also by the same transformation, for $|\mu| < r$,

$$Z(\alpha,\, k_1) = \frac{2}{1 + r^2}\left[Z(\beta, r^2) + \frac{\mu((1 - \mu^2)(r^4 - \mu^2))^{1/2}}{r^2 + \mu^2}\right],$$

where

$$\operatorname{sn}\,(\beta,\, r^2) = \mu/r^2;$$

whereas for $|\mu| > r$

$$Z(\alpha,\, k_1) = \frac{2}{1 + r^2}\left[Z(\beta', r^2) + \frac{r^2((\mu^2 - 1)(\mu^2 - r^4))^{1/2}}{\mu(\mu^2 + r^2)}\right],$$

where

$$\operatorname{sn}\,(\beta',\, r^2) = 1/\mu.$$

Consequently these equations, together with (14), (21), (23), and (25), give

$$\frac{r^2}{\pi i}\int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1 - \mu}\,d\sigma_1 = \begin{cases} M - (1-\mu^2)^{1/2} - \dfrac{2K}{\pi}\,\dfrac{((1-\mu^2)(r^4-\mu^2))^{1/2}}{\mu}\,Z(\beta+K,\, r^2) & \text{for } |\mu| < r, \\[2ex] -M + (1 - r^4/\mu^2)^{1/2} + \dfrac{2K}{\pi}\,\dfrac{((\mu^2-1)(\mu^2-r^4))^{1/2}}{\mu}\,Z(\beta'+K,\, r^2) & \text{for } |\mu| > r, \end{cases} \tag{26}$$

where

$$M = \frac{1}{\pi}\,[\pi + 2(1 - r^4)K - 2E].$$

In view of the inequalities in (12) in which $p_n = r^{2n}$, equations (26) give, for $r < |\zeta| < 1$,

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[11] Cf. H. Hancock, *Theory of elliptic functions*, vol. 1, New York, 1910, p. 250.

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\frac{r^2}{\pi i}\int_{\gamma_1}\frac{u_1(\sigma_1)}{\sigma_1-p_n\zeta}\,d\sigma_1
$$

$$
(27)\qquad = \begin{cases} M-(1-p_n^2\zeta^2)^{1/2}-\dfrac{2K}{\pi}\,\dfrac{((1-p_{n-1}^2\zeta^2)(1-p_n^2\zeta^2))^{1/2}}{p_{n-1}\zeta}Z(\beta_n+K,\,r^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } n\geqq 1, \\ -M+(1-p_{1-n}^2/\zeta^2)^{1/2}+\dfrac{2K}{\pi}\,\dfrac{((1-p_{-n}^2/\zeta^2)(1-p_{1-n}^2/\zeta^2))^{1/2}}{p_{-n}/\zeta}Z(\beta'_{-n}+K,\,r^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } n<1, \end{cases}
$$

where

$$
\operatorname{sn}(\beta_n,\,r^2) = p_{n-1}\zeta \quad\text{and}\quad \operatorname{sn}(\beta'_n,\,r^2) = p_n/\zeta.
$$

If $\sigma_2 = e^{i\phi}$ and γ_2 represents the circle $|\zeta| = 1$, it follows easily from (21) and (22), on setting $r = 1$, that

$$
\frac{1}{2\pi i}\int_{\gamma_2}\frac{\omega(\sigma_2)\overline{\omega(\sigma_2)}}{\sigma_2-\mu}\,d\sigma_2 = \frac{2}{\pi}+\frac{1-\mu^2}{2\pi\mu}\log\left(\frac{1+\mu}{1-\mu}\right)^2+\begin{cases}-(1-\mu^2)^{1/2} & \text{for } |\mu|<1, \\ (1-1/\mu^2)^{1/2}-1 & \text{for } |\mu|>1.\end{cases}
$$

The constant c_2 in the definition of the function $u_2(\sigma_2)$ in (14) is therefore

$$
c_2 = \frac{1}{2} - \frac{2}{\pi}.
$$

Consequently these equations, together with the inequalities in (12), give, for $r<|\zeta|<1$,

$$
\frac{1}{\pi i}\int_{\gamma_2}\frac{u_2(\sigma_2)}{\sigma_2 - p_n\zeta}\,d\sigma_2
$$

$$
(28)\qquad = \begin{cases} 1-\dfrac{2}{\pi}-(1-p_n^2\zeta^2)^{1/2}+\dfrac{1-p_n^2\zeta^2}{\pi p_n\zeta}\log\dfrac{1+p_n\zeta}{1-p_n\zeta} & \text{for } n \geqq 0, \\ -1+\dfrac{2}{\pi}+(1-p_{-n}^2/\zeta^2)^{1/2}-\dfrac{1-p_{-n}^2/\zeta^2}{\pi p_{-n}/\zeta}\log\dfrac{1+p_{-n}/\zeta}{1-p_{-n}/\zeta} & \text{for } n > 0. \end{cases}
$$

Thus, by equations (10), (27), and (28), the function $f(\zeta)$ which is regular for $r<|\zeta|<1$ and satisfies the condition (14) on the circles γ_1 and γ_2 can, after proper rearrangement of terms, be written in the form

$$
(29)\qquad \begin{aligned} f(\zeta) = {} & -(1-\zeta^2)^{1/2}+\frac{1-\zeta^2}{\pi\zeta}\log\frac{1+\zeta}{1-\zeta} \\ & + \sum_{n=1}^{\infty}\left[T_n(\zeta) - T_n(1/\zeta)\right] + \text{const.}, \end{aligned}
$$

where

$$T_n(\zeta) = \frac{1 - r^2}{r}(1 - r^{4n}\zeta^2)^{1/2}$$
$$+ \frac{1 - r^{4n}\zeta^2}{\pi r^{2n}\zeta}\left[\log\frac{1 + r^{2n}\zeta}{1 - r^{2n}\zeta} + 2K\left(\frac{1 - r^{4n-4}\zeta^2}{1 - r^{4n}\zeta^2}\right)^{1/2} Z(\beta_n + K, r^2)\right]$$

and

$$\operatorname{sn}(\beta_n, r^2) = r^{2n-2}\zeta.$$

Since $\operatorname{sn}(0, r^2) = 0$ and $Z(K, r^2) = 0$, it is at once evident that

$$\lim_{r\to 0}\,[T_n(\zeta) - T_n(1/\zeta)] = 0$$

for $n \geqq 1$. Consequently as r tends to zero the function $f(\zeta)$ defined in (29) reduces to the solution of the torsion problem for the simply connected region bounded by two circular arcs of equal radii and intersecting at right angles at the points $z = \pm 1$ [12].

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(12) Cf. Sokolnikoff and Soknolnikoff [7, p. 386].

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THE UNIVERSITY OF MICHIGAN,
ANN ARBOR, MICH.